August Qualifying Exam

MTH 849: PDE2 *(Spring 2022)* **Willie WY Wong**

Instructions

The qualifying exam is closed-book, no outside resources or collaboration. A passing grade roughly corresponds to providing complete and correct answers to four (or more) of the following seven questions in the time alloted.

You may refer to facts proven both in the course and in your problem sets as "known facts" (provided that you state them correctly) without justification. Some additional useful reminders can be found below.

Convenient Reminders

- Hölder's inequality: $|\int fg| \leq \|f\|_{L^p}\|g\|_{L^q}$ if $p^{-1} + q^{-1} = 1$.
- Lebesgue interpolation: $\|u\|_{L^p} \leq \|u\|_{L^q}^{\theta}\|u\|_{L^r}^{1-\theta}$ if $q < p < r$ and $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$.
- Young's inequality: given $a, b \in \mathbb{R}_+$ and $p^{-1} + q^{-1} = 1$, we have $ab \leq \frac{1}{2}((\varepsilon a)^p + (b/\varepsilon)^q)$ for all $\varepsilon > 0$.
- Sobolev embedding exponents: starting from $W^{1,p}$ on a domain in $\mathbb{R}^d$, it embeds into:
  - L^q if $p < d$, where $q \in [p, \frac{dp}{d-p}]$;
  - $C^{0,\alpha}$ if $p > d$, where $\alpha \in (0, 1 - d/p]$.
- Notation: $B(x,r)$ is the (open) ball of radius r centered at x.
- Useful Lemma (from Cantor Diagonalization Worksheet): Let $S \subset X$ be fixed, and a sequence of functions $f_k : S \to X$ be given, such that $d(f_k(x), x) < \frac{1}{k}$ for every $x \in S$. If the images $f_k(S)$ are all Cauchy precompact, then S is Cauchy precompact.
- Lax-Milgram: If $B : H \times H \to \mathbb{R}$ is a bilinear form defined on a Hilbert space H, for which
  - boundedness: $\exists \alpha > 0$ such that $|B[u,v]| \leq \alpha\|u\|\|v\|$;
  - coercivity: $\exists \beta > 0$ such that $\beta\|u\|^2 \leq B[u,u]$

  holds. Then for any bounded linear functional $f \in H^*$, there exists a unique element $u \in H$ such that $B[u,v] = f(v)$ for all $v \in H$.
- Rayleigh's formula: given a symmetric, uniformly elliptic operator L, the principal eigenvalue satisfies $\lambda_1 = \min\{B[u,u]/\|u\|_{L^2}^2 : u \in H_0^1 \setminus \{0\}\}$.
- Spectral Theorem: if $Lu = -\sum \partial_i(a_{ij}\partial_j u)$ is uniformly elliptic, then
  - Eigenfunctions of L form a basis of L^2, and can be ordered with eigenvalues increasing (counted with multiplicity).
  - Principal eigenvalue is positive. If domain is connected, principal eigenvalue is simple, with signed eigenfunction.
  - There are finitely many eigenvalues in each bounded interval.

Problem 1. Consider the thin plate equation $\partial^2_{tt} u = -\Delta\Delta u$ for $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$.

1. Verify that if u is a $C^4(\mathbb{R}\times\mathbb{R}^2)$ solution to the thin plate equation, then so is $u_\lambda(t,x) := u(\lambda^{-2}t, \lambda^{-1}x)$; here $\lambda \in (0,\infty)$.
2. Consider the functional
$$E[u] = \int_{\mathbb{R}^2} |\partial_t u(0,x)| + |\Delta u(0,x)|\, dx.$$
How are $E[u]$ and $E[u_\lambda]$ related (where u_λ is defined as in the previous part)?
3. Consider the following statement:

   > There exist a constant C such that every $C^4(\mathbb{R}\times\mathbb{R}^2)$ solution u of the thin plate equation satisfies $\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^2)} \le CE[u]$.

   Prove that the statement is *false* for every $p \in [1,\infty)$.
   Hint: choose a suitable sequence λ_j and apply scaling.

Problem 2. Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Prove that there exists some $M > 0$ such that whenever $u \in W^{1,3}(\Omega)$ satisfies $u(x,y) = -u(x,-y)$ and $\|\nabla u\|_{L^3(\Omega)} \le 1$, we have then $|u(x,y)| \le M|y|^{1/3}$ almost everywhere.

Problem 3. Let Ω be a bounded connected open domain in $\mathbb{R}^d$ with C^1 boundary. Fix $p \in [1,\infty)$. Let $G : W^{1,p}(\Omega) \to \mathbb{R}$ be a continuous function that satisfies:

$$G(\lambda u) = \lambda G(u) \qquad \forall u \in W^{1,p}(\Omega), \lambda \in \mathbb{R} \tag{3.1}$$
$$G(\mathbf{1}) \neq 0, \text{ where } \mathbf{1} \text{ is the constant function equalling 1 everywhere} \tag{3.2}$$

Prove: there exists $C > 0$ such that whenever $u \in W^{1,p}(\Omega)$ satisfies $G(u) = 0$, then

$$\|u\|_{L^p(\Omega)} \le C\|Du\|_{L^p(\Omega)}$$

Hint: argue by contradiction and use compactness.

Problem 4. Let Ω_A, Ω_B be two bounded open sets with C^1 boundaries, such that $\Omega_A \subsetneq \Omega_B$. Let L_A and L_B be divergence form operators

$$L_A u = -\sum \partial_i(a_{ij}\partial_j u), \qquad L_B u = -\sum \partial_i(b_{ij}\partial_j u),$$

where the symmetric matrix-valued functions a_{ij} and b_{ij} are smooth functions on Ω_B and uniformly elliptic. Given:

- The principal eigenvalue of L_B on the domain Ω_B is λ_B.
- There exists a number $\kappa > 0$ such that at every point in Ω_A, the matrix $\kappa a_{ij} - b_{ij}$ is positive semidefinite.

Provide a (strictly positive) lower bound for the principal eigenvalue of L_A on the domain Ω_A based on the information given; fully justify your answer.

(questions continue on next page)

Problem 5. Let Ω be a bounded open domain in $\mathbb{R}^2$ with C^1 boundary. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous, and satisfy

$$\limsup_{s\to-\infty} f(s) < 0 < \liminf_{s\to+\infty} f(s).$$

Prove: there exists $M > 0$ (depending only on f) such that whenever $u \in C^2(\bar{\Omega})$ solves

$$\partial_x[(\partial_x u)^2] + \partial^2_{yy} u = f(u)$$

then we have

$$\sup_{\Omega} |u| \leq \max(M, \sup_{\partial\Omega} |u|).$$

Problem 6. Let Ω be a bounded open domain, and L a symmetric uniformly elliptic operator in divergence form, with C^∞ coefficients. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ be an increasing enumeration of the eigenvalues of L counted with multiplicity.

Given $\lambda_1 = 1$ and $\lambda_2 = 3$, consider the problem

$$Lu - 2u = f, \qquad u|_{\partial\Omega} = 0.$$

Prove: for every $f \in L^2$, there exists a unique solution $u \in H^1_0(\Omega)$ to the problem above. Furthermore, show that the solution satisfies $\|u\|_{L^2} \leq \|f\|_{L^2}$.

Problem 7. Let $S = \{u \in H^1(\mathbb{R}^d) : \int |\nabla u|^2 + |x|^2|u|^2 \, dx \leq 1\}$. Prove that S is Cauchy precompact in L^2.

Hints: use the "useful lemma" from page 1. For the sequence of functions f_k, let $f_k(u) = \chi_k u$, where the cutoff function χ_k is smooth, has compact support on $B(0,2k)$ and is equal to the identity on $B(0,k)$.